# The parametrisation method for invariant manifolds: application to Hopf bifurcations in follower force problems

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Séminaires internes pôle mécanique
September 2025

# Scope of this presentation

**Nonlinear vibrating structures:**

- Distributed smooth (geometric) nonlinearities
- Large vibration amplitudes
- Reduced-order models (ROMs)
- Simulation-free (data-free) ROMs
- FEM models

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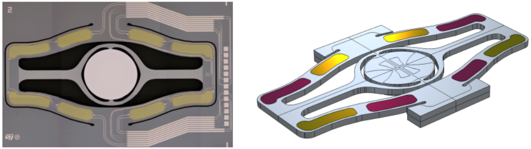

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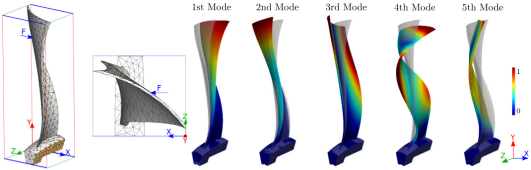


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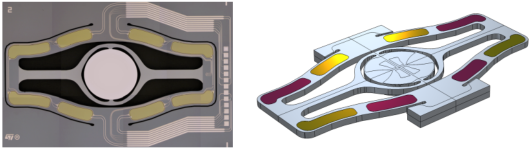
**Why reduce?**

- Faster computations
- More interpretable models
- General results and possibility of analytical solutions



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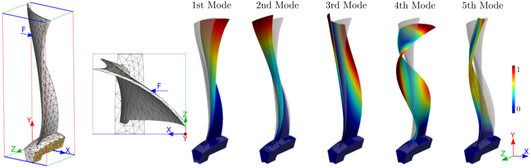


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- Linear vibration modes
- Modal derivatives
- Dual modes
- Proper orthogonal decomposition (POD)
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**Fully nonlinear relationship between the master and slave coordinates**

# Linear vibration modes - Geometric perspective

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In order to reduce the model we gather **some** (how many?) of the modes in a matrix $\boldsymbol{\Phi}$ and impose

$$\mathbf{U} = \boldsymbol{\Phi}\mathbf{z}$$

to transform the equations into

$$\ddot{\mathbf{z}} + \boldsymbol{\Lambda}^2 \mathbf{z} = \mathbf{0}$$

# Linear vibration modes - Geometric perspective

We will take an alternative (dynamical systems) approach:

$$\mathbf{B}\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$$

with

$$\mathbf{y} = \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ -\mathbf{K} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}$$

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**Nice geometric interpretation: each pair of eigenvalues defines an invariant subspace in phase space!**

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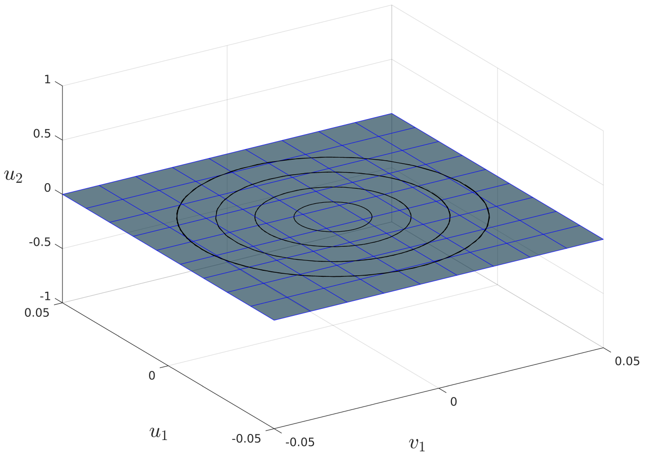
$$\ddot{u}_1 + \omega_1^2 u_1 = 0$$
$$\ddot{u}_2 + \omega_2^2 u_2 = 0$$



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What happens when we add nonlinearities?

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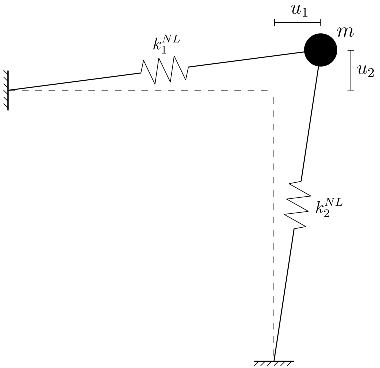
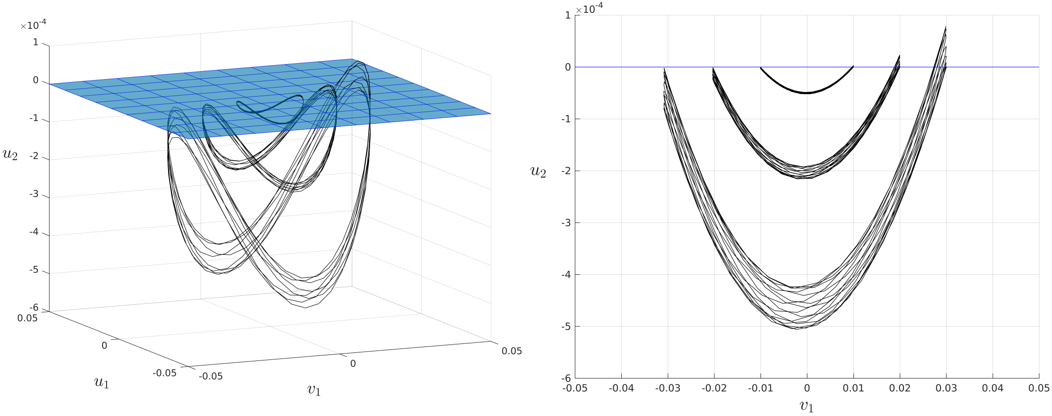



$$\ddot{u}_1 + \omega_1^2 u_1 + \frac{\omega_1^2}{2}\left(3u_1^2 + u_2^2\right) + \omega_2^2 u_1 u_2 + \frac{\omega_1^2+\omega_2^2}{2} u_1\left(u_1^2+u_2^2\right) = 0$$

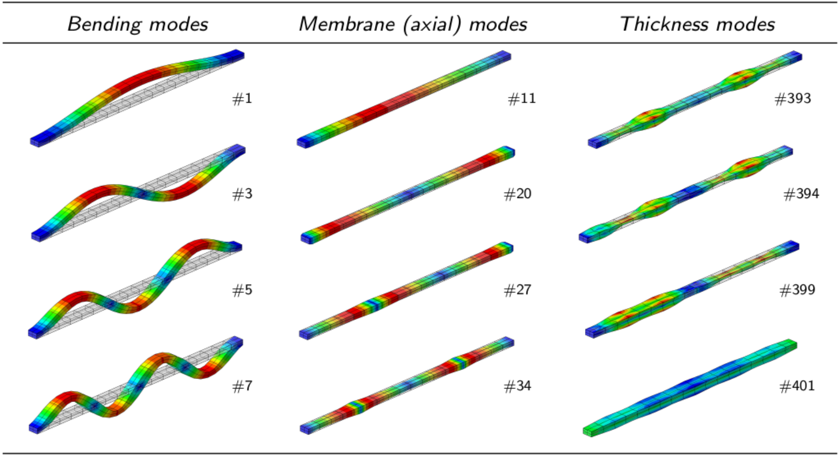
$$\ddot{u}_2 + \omega_2^2 u_2 + \frac{\omega_2^2}{2}\left(3u_2^2 + u_1^2\right) + \omega_1^2 u_1 u_2 + \frac{\omega_1^2+\omega_2^2}{2} u_2\left(u_1^2+u_2^2\right) = 0$$

We fix $\omega_1 = 1$ and $\omega_2 = \sqrt{2}$

# Linear vibration modes - Nonlinear problems



# An illustrative example - Clamped-clamped 3D FE beam [Vizzaccaro et al. (2020)]



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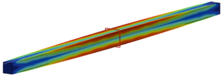
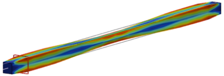
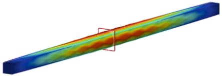
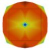
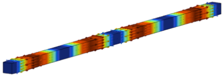
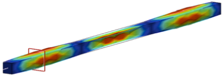
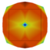
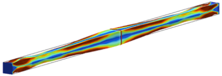
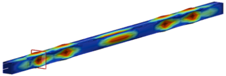
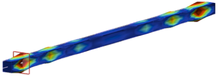
| # | $\omega_l/(2\pi)[Hz]$ | $\mathcal{C}^{1l}_{111}/\Gamma^{1}_{111}$ | Coupled modes Shape | Section |
|---|---|---|---|---|
| 330 | 1.137e5 | 3.22e-1 |  |  |
| 328 | 1.125e5 | 1.34e-1 |  |  |
| 1143 | 3.698e5 | 1.05e-1 |  |  |
| 34 | 1.043e4 | 9.26e-2 |  |  |
| 1147 | 3.699e5 | 7.01e-2 |  |  |
| 370 | 1.202e5 | 4.60e-2 |  |  |
| 167 | 7.644e4 | 3.35e-2 |  |  |
| 172 | 7.653e4 | 2.93e-2 |  |  |

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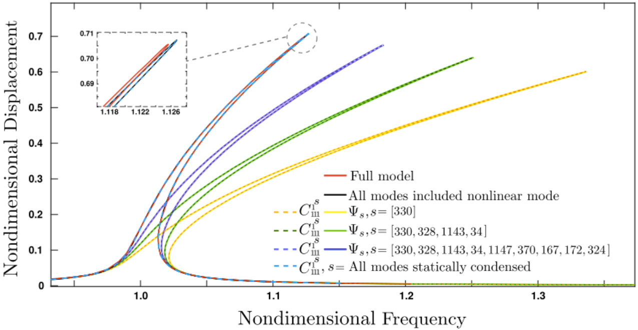


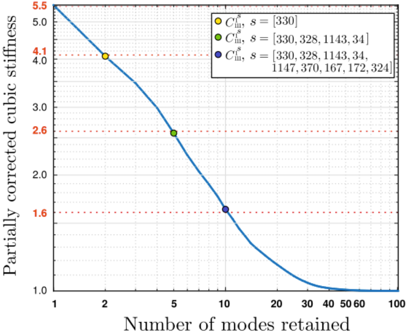

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How to compute?

- Center manifold theory [Shaw and Pierre (1991, 1993, 1994)]
- Normal form technique [Jézéquel and Lamarque (1991); Touzé (2003); Touzé et al. (2004); Touzé and Amabili (2006)]
- Parametrisation method for invariant manifolds [Cabré et al. (2003a,b, 2005); Haro et al. (2016)]

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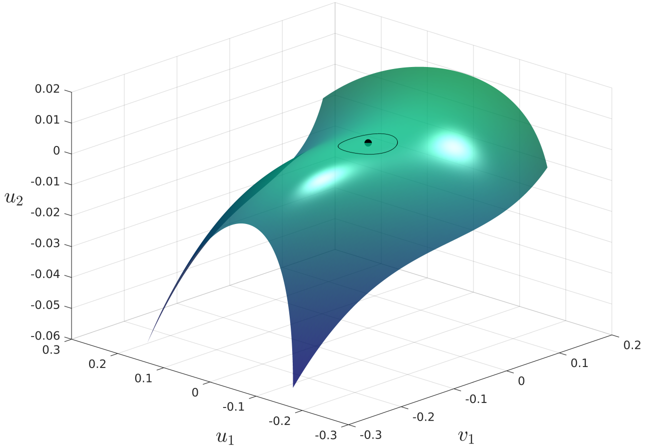

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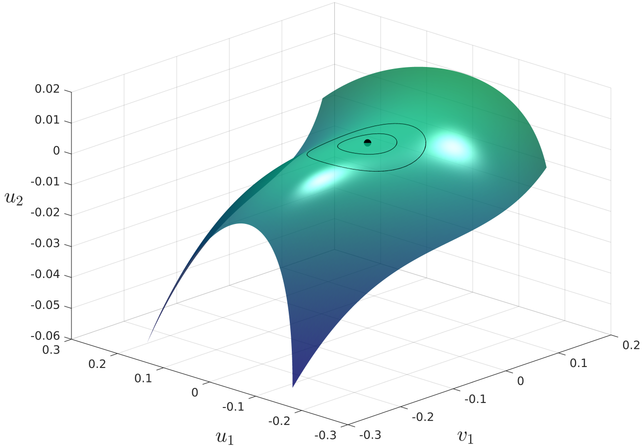

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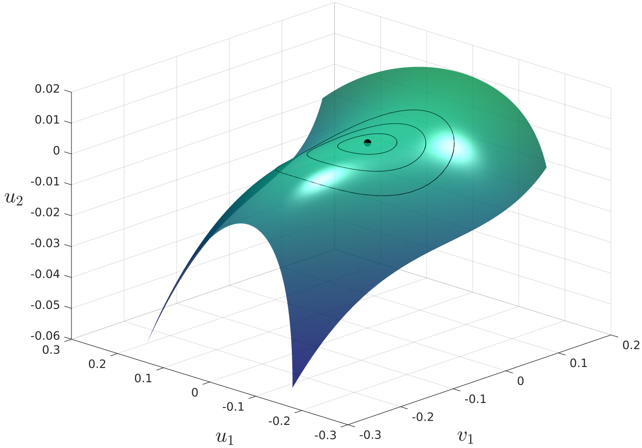

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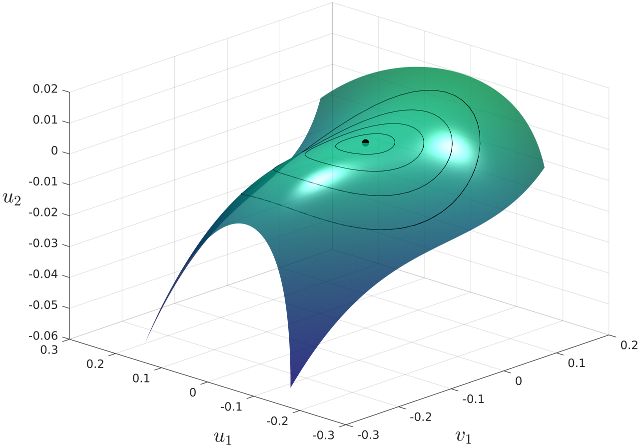

# Direct parametrisation of invariant manifolds [Vizzaccaro et al. (2024)]

We will consider mechanical systems of the form

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} + \mathbf{G}(\mathbf{U}, \mathbf{U}) + \mathbf{H}(\mathbf{U}, \mathbf{U}, \mathbf{U}) = \mathbf{F}(t)$$

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$$\mathbf{y} = \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \\ \mathbf{R} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{M} & \mathbf{0} \\ -\mathbf{K} & \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

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**To compute the manifold we introduce new (normal) coordinates $\mathbf{z}$**

$$\mathbf{z} \in \mathbf{C}^d, \quad \mathbf{y} \in \mathbf{R}^D, \quad d \ll D$$

# Direct parametrisation of invariant manifolds [Vizzaccaro et al. (2024)]

In order to compute the manifold we introduce its parametrisation and reduced dynamics

$$\mathbf{y} = \mathbf{W}(\mathbf{z})$$
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To fulfill the invariance property they must verify the invariance equation

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$$\mathbf{y} = \mathbf{W}(\mathbf{z}) = \sum_{p=1}^{o} [\mathbf{W}(\mathbf{z})]_p = \sum_{p=1}^{o} \sum_{k=1}^{m_p} \mathbf{W}^{(p,k)} \mathbf{z}^{\boldsymbol{\alpha}(p,k)}$$

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}) = \sum_{p=1}^{o} [\mathbf{f}(\mathbf{z})]_p = \sum_{p=1}^{o} \sum_{k=1}^{m_p} \mathbf{f}^{(p,k)} \mathbf{z}^{\boldsymbol{\alpha}(p,k)}$$

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Which is solved order-by-order $\forall p \in \{1, \dots, o\}$:

$$[\mathbf{B}\nabla_{\mathbf{z}}\mathbf{W}(\mathbf{z})\mathbf{f}(\mathbf{z})]_p = [\mathbf{A}\mathbf{W}(\mathbf{z})]_p + [\mathbf{Q}(\mathbf{W}(\mathbf{z}), \mathbf{W}(\mathbf{z}))]_p.$$

# Direct parametrisation of invariant manifolds [Vizzaccaro et al. (2024)]

We solve first the order 1 equation. We note that

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In what follows we also need to define the left eigenvalue problem:

$$\mathbf{X}_s^*(\lambda_s\mathbf{B} - \mathbf{A}) = \mathbf{0}$$

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The homological equation at order $p$ is

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For monomial $(p,k)$:

$$(\sigma^{(p,k)}\mathbf{B} - \mathbf{A})\mathbf{W}^{(p,k)} + \sum_{s=1}^{d} \mathbf{B}\mathbf{Y}_s f_s^{(p,k)} = \mathbf{R}^{(p,k)}$$

with the $\mathbf{R}^{(p,k)}$ computed only from the previous orders and

$$\sigma^{(p,k)} = \sum_{s=1}^{d} \boldsymbol{\alpha}(p,k)_s \cdot \lambda_s$$

# Direct parametrisation of invariant manifolds [Vizzaccaro et al. (2024)]

The homological equation at order $p$ is

$$\mathbf{B}[\nabla_{\mathbf{z}}\mathbf{W}(\mathbf{z})\mathbf{f}(\mathbf{z})]_p = \mathbf{A}[\mathbf{W}(\mathbf{z})]_p + [\mathbf{Q}(\mathbf{z},\mathbf{z})]_p$$

For monomial $(p,k)$:

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**Problem: too many unknowns!**

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To find a solution, we project into the modal basis:

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The condition $\xi_s^{(p,k)} = 0$ translates into physical space as

$$\mathbf{X}_s^* \mathbf{B} \mathbf{W}^{(p,k)} = 0$$

# Direct parametrisation of invariant manifolds [Vizzaccaro et al. (2024)]

Finally, for each monomial a homological equation

$$\begin{bmatrix} \sigma^{(p,k)}\mathbf{B}-\mathbf{A} & \mathbf{B}\mathbf{Y}_{\mathcal{R}} & \mathbf{0} \\ \mathbf{X}_{\mathcal{R}}^{\star}\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{W}^{(p,k)} \\ \mathbf{f}_{\mathcal{R}}^{(p,k)} \\ \mathbf{f}_{\not\mathcal{R}}^{(p,k)} \end{bmatrix} = \begin{bmatrix} \mathbf{R}^{(p,k)} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

is solved in order to find the unknown coefficients $\mathbf{W}^{(p,k)}$ and $\mathbf{f}^{(p,k)}$.

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**The system can be treated as in the autonomous case!**

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**Outline of this part**

- Inclusion of the bifurcation parameter
- Ziegler's pendulum
  - Linear stability analysis
  - Master mode selection
  - Results
- Beck's column (FE model)
- Conclusions
- A similar example: NS equations

# Adding the bifurcation parameter

We consider problems of the form

$$\underbrace{\begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}}_{\widetilde{\mathbf{B}}} \underbrace{\begin{bmatrix} \dot{\mathbf{y}} \\ \mu \end{bmatrix}}_{\dot{\widetilde{\mathbf{y}}}} = \underbrace{\begin{bmatrix} \mathbf{A}_t & \mathbf{A}_0 \\ \mathbf{0} & 0 \end{bmatrix}}_{\widetilde{\mathbf{A}}_t} \underbrace{\begin{bmatrix} \mathbf{y} \\ \mu \end{bmatrix}}_{\widetilde{\mathbf{y}}} + \underbrace{\begin{bmatrix} \mathbf{Q}_1(\mathbf{y}, \mathbf{y}) + \mathbf{Q}_2(\mathbf{y}, \mu) + \mathbf{Q}_3(\mu, \mu) \\ 0 \end{bmatrix}}_{\widetilde{\mathbf{Q}}(\widetilde{\mathbf{y}}, \widetilde{\mathbf{y}})}$$

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Already in **normal form:**

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And of **trivial reduced dynamics:**

$$\dot{\widetilde{\mathbf{z}}} = \mathbf{f}(\widetilde{\mathbf{z}}), \quad \text{with} \quad f_{d+1}(\widetilde{\mathbf{z}}) = 0$$

## Adding the bifurcation parameter

The parametrisation and reduced dynamics are expanded in polynomial form:

$$\mathbf{W}(\widetilde{\mathbf{z}}) = \sum_{p=1}^{o} [\mathbf{W}(\widetilde{\mathbf{z}})]_p = \sum_{p=1}^{o} \sum_{k=1}^{m_p} \mathbf{W}^{(p,k)} \widetilde{\mathbf{z}}^{\boldsymbol{\alpha}(p,k)}$$

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# Ziegler's pendulum [Ziegler (1952)]

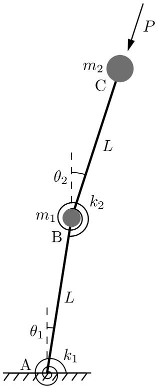
The equations of motion are [Luongo and D'Annibale (2015)] :

$$\mathbf{M}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}} + (\mathbf{K} + \mathbf{K_g})\,\boldsymbol{\theta} = \mathbf{F}_{nl}$$

with $\mathbf{C} = 2\left(\xi_m \mathbf{M} + \xi_k \mathbf{K}\right)$ and

$$\mathbf{M} = L^2 \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$\mathbf{K_g} = PL \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{F}_{nl} = -\frac{PL}{6} \begin{bmatrix} (\theta_1 - \theta_2)^3 \\ 0 \end{bmatrix}, \quad \boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}.$$



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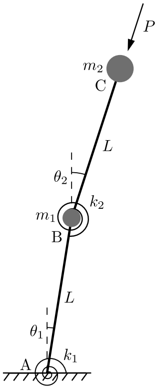
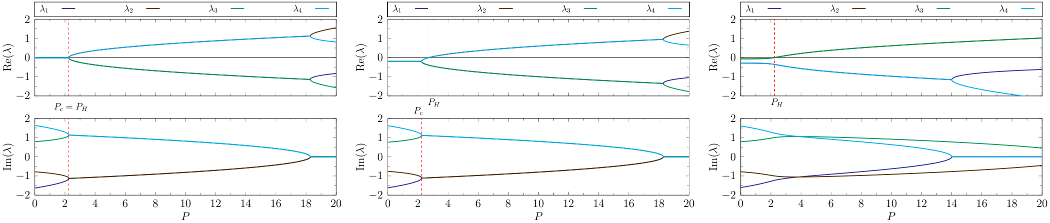



The numerical values of the parameters are chosen as

$$k_1 = \delta^2 k_2, \quad m_1 = \gamma^2 m_2, \quad k_2 = m_2 = 1,$$

$$\delta^2 = \frac{41}{4}, \quad \gamma^2 = \frac{25}{4}.$$

# Linear stability and master modes




(a) Conservative system (b) $\xi_m = 0.2$ and $\xi_k = 0$ (c) $\xi_m = 0$ and $\xi_k = 0.1$

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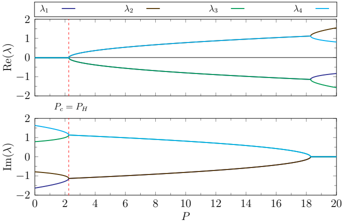


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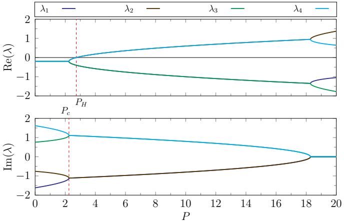


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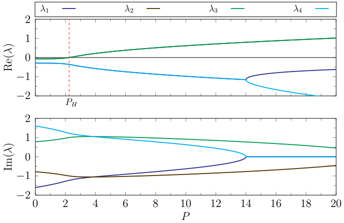


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- Conservative system: two eigenfrequencies coalesce at the bifurcation.

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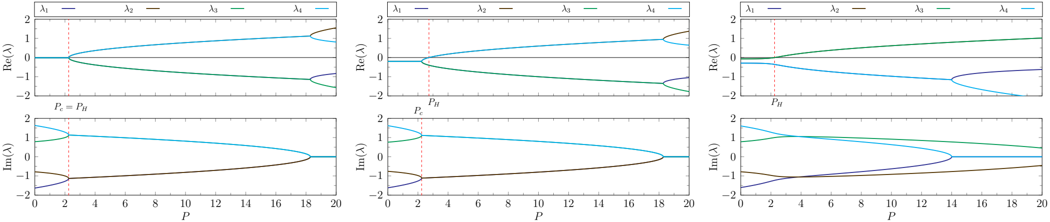


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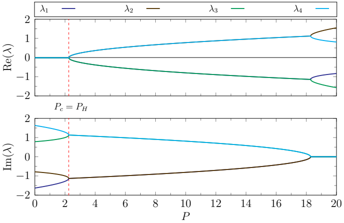


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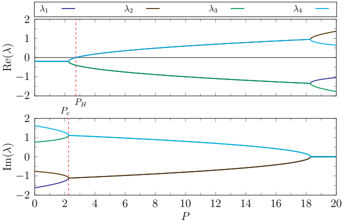


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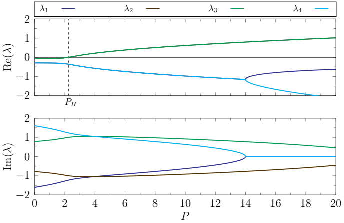


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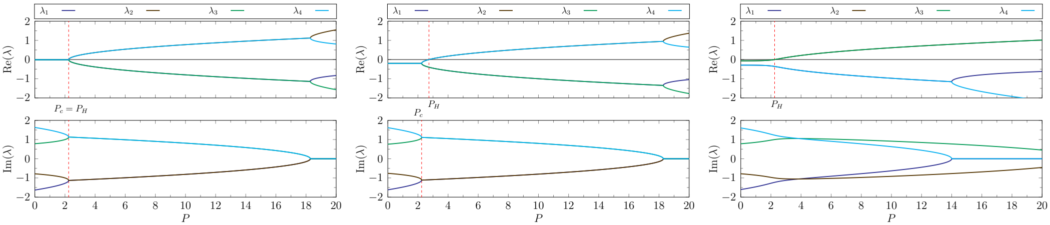


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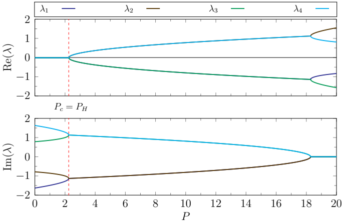

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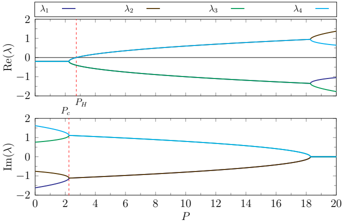

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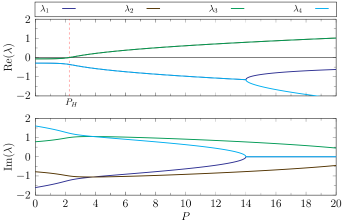

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- Another choice: only keeping the unstable mode [Li and Wang (2024)].

## Bifurcation diagrams - Mass proportional damping

- Objective: construct a **single ROM**, at an expansion point $P_e$, and use it to trace the bifurcation diagram of the system.

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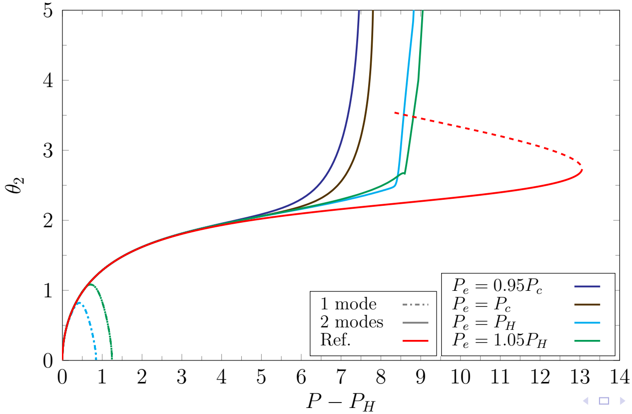
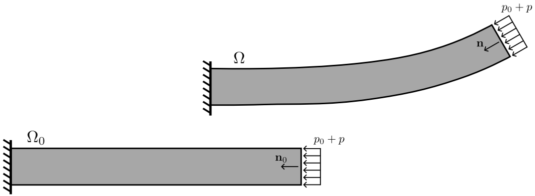


# Beck's column




- Column subjected to a follower force
- Plane stress finite element model
- $\sim 600$ degrees of freedom

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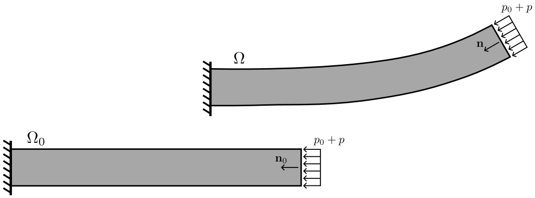


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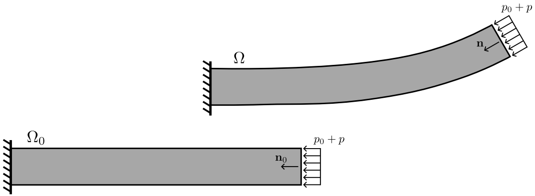


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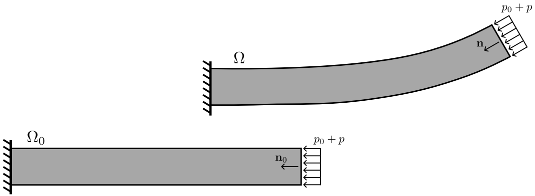


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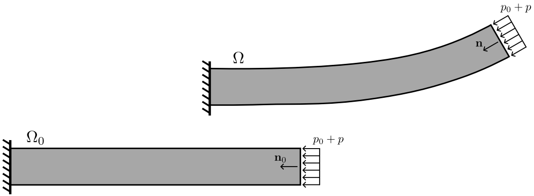
$$\delta\mathcal{P}_{ext} = \int_{\partial\Omega_0} \tilde{\mathbf{v}} \cdot (p_0 + p)\left(\mathbf{n}_0 + \frac{\mathbf{e}_3 \times \mathbf{u}_{,a}}{J_{s0}}\right) \mathrm{d}s_0$$

$$\int_{\partial\Omega_0} \tilde{\mathbf{v}} \cdot \mathbf{n}_0 \, \mathrm{d}s_0 = \tilde{\mathbf{V}}^T \int_{\hat{\Omega}_e} \mathbf{N}^T \mathbf{E}_3 \mathbf{N}_{,a} \mathbf{X} \, \mathrm{d}a = \tilde{\mathbf{V}}^T \mathbf{R}_0^e$$

$$\int_{\partial\Omega_0} \frac{\mathbf{e}_3 \times \mathbf{u}_{,a}}{J_{s0}} \mathrm{d}s_0 = \tilde{\mathbf{V}}^T \left(\int_{\hat{\Omega}_e} \mathbf{N}^T \mathbf{E}_3 \mathbf{N}_{,a} \mathrm{d}a\right) \mathbf{U} = \tilde{\mathbf{V}}^T \mathbf{R}_u^e \mathbf{U}$$

# Beck's column

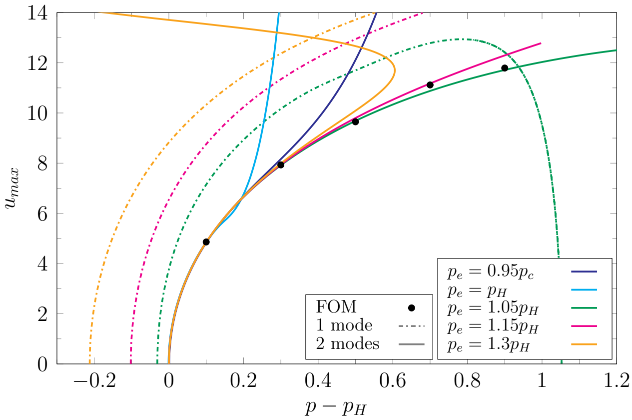



- Column subjected to a follower force
- Plane stress finite element model
- $\sim 600$ degrees of freedom

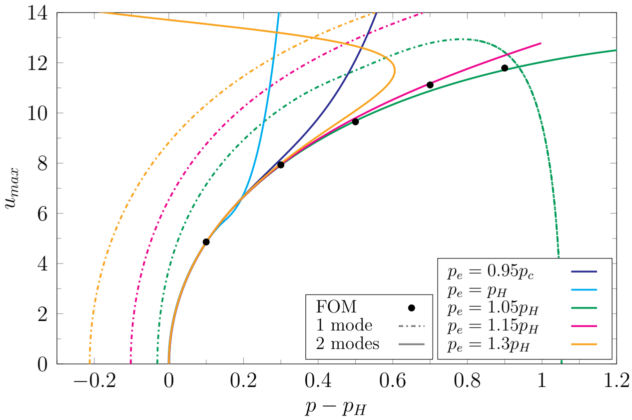
$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}_t\mathbf{U} - p\mathbf{R}_t + \mathbf{G}_t(\mathbf{U},\mathbf{U}) - p\mathbf{R}_u\mathbf{U} + \mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U}) = \mathbf{0}$$

For further details, see [Vizzaccaro et al. (2024); Stabile et al. (2025)].

# Beck's column

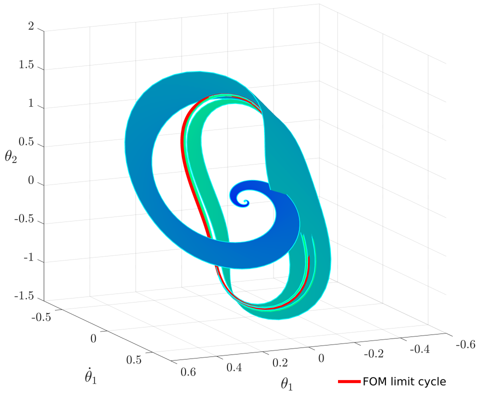


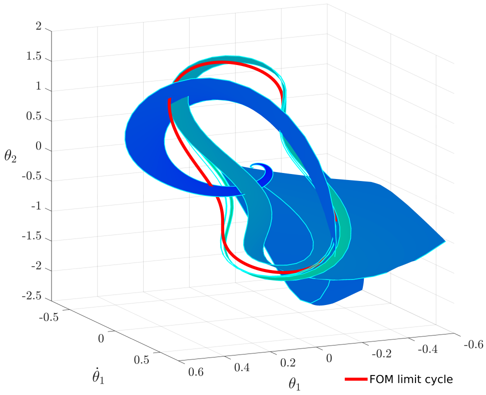
# Beck's column




**Parametrising the unstable manifold yields better results!**

# Phase space interpretation



## Conclusion

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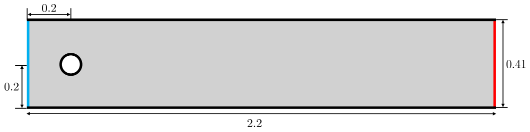
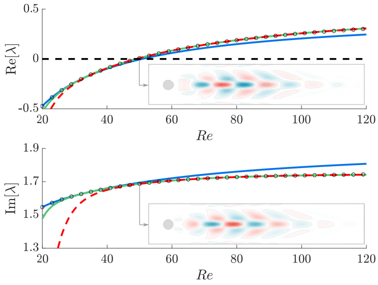
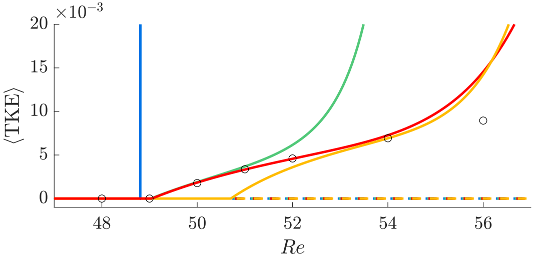
- It is possible to trace bifurcation diagrams with ROMs constructed at a single parameter value.
- The approach remains valid for a range of parameters considerably larger than the single mode strategy.
- Parametrising after the bifurcation yields better results.

# Bonus - Navier-Stokes equations [Colombo et al. (2025), submitted]



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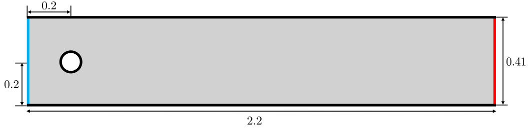


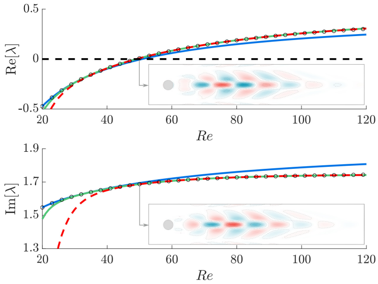


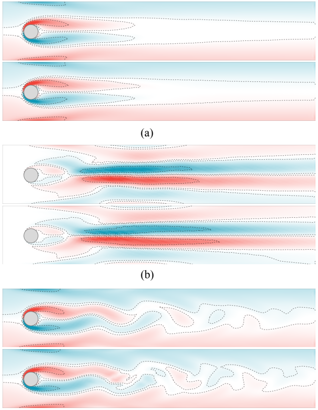

(a)

(b)

(c)

# THANK YOU

## FOR YOUR ATTENTION

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